# M E T U
# Department of Mathematics

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| | **Discrete Mathematics**<br>**MidTerm I** | |
| Code : *Math 112*<br>Acad. Year : *2015-2016*<br>Semester : *Spring*<br>Instructor : *M.Bhupal, S.Finashin, A.Doğanaksoy, S.Önal*<br>Date : *1.04.2016*<br>Time : *17.40*<br>Duration : *100 minutes* | Last Name :<br>Name :<br>Department :<br>Signature : | Student No : |
| | **5 Questions on 4 Pages**<br>**Total 60 Points** | |

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READ THE PROBLEMS CAREFULLY AND GIVE DETAILED WORK

**1. (5+5+5 pts)** Let $S = \{a, b, c, d, e, f, 0, 1, 2, 3, \ldots, 9\}$ (6 letters and 10 digits).

**a)** Find the number of permutations of $S$ in which all six letters appear before digits.

$6! \cdot 10!$

**b)** Find the number of permutations containing no consecutive pairs of letters.

$10! \cdot P(11, 6) = 10! \cdot \binom{11}{6} \cdot 6!$

permute the digits.

we have 11 places for 6 letters.

**c)** Find the number of arrangements of the elements of $S$ in four consecutive rows (some of the rows are allowed to be empty).

$$\frac{(16+3)!}{3! \cdot (1!)^{16}}$$

**2. (7+8 pts)**

**a)** In how many ways one can distribute 10 blue and 10 white pencils to 5 children in such a way that each child receives at least one pencil of each color ?

Number of ways of distributing blue pencils is

$$\binom{5+5-1}{5-1} = \binom{9}{4}$$

Same for white pencils.

So answer is $\binom{9}{4} \cdot \binom{9}{4} = 126^2$.

**b)** In how many ways is it possible to split 7 boys and 5 girls into two groups of equal sizes so that each group contains at least one boy and at least one girl ?

4 girls + 2 boys and 1 girl + 5 boys : $\binom{5}{4} \cdot \binom{7}{2} = 105$

3 girls + 3 boys and 2 girls + 4 boys : $\binom{5}{3} \cdot \binom{7}{3} = 350$

Answer : $105 + 350 = 455$

**3. (5+5+5 pts)** An expansion presents $(x+2y-3z+4)^{15}$ as a sum of monomials of the form $c_{ijk}x^iy^jz^k$, where $c_{ijk}$ is a coefficient.

**a)** Find the coefficient $c_{143}$ at $xy^4z^3$.

$$\frac{15!}{4!\,3!\,7!}\cdot x\cdot(2y)^4\cdot(-3z)^3\cdot(4)^7$$

$$=\underbrace{\frac{15!}{4!\cdot 3!\cdot 7!}\cdot 2^4\cdot(-3)^3\cdot 4^7}_{\text{coefficient } c_{143}}\cdot xy^4z^3$$

**b)** Find the total number of monomials $c_{ijk}x^iy^jz^k$ in the expansion.

The number of monomials = the number of nonnegative integer solutions of $n_1+n_2+n_3+n_4=15$

i.e. $\binom{15+4-1}{4-1}=\binom{18}{3}$

**c)** Find the sum of all of the coefficients $c_{ijk}$.

Put $x=y=z=1$ to find the sum of the coefficients.

$(1+2-3+4)^{15}=4^{15}$.

**4. (7 pts)** Find the coefficient at $x^{20}$ after expansion of $(x^3 + x^4 + \ldots + x^9)^4$.

$$\underbrace{(x^3 + \cdots + x^9)}_{x^{a_1}}\underbrace{(x^3 + \cdots + x^9)}_{x^{a_2}}\underbrace{(x^3 + \cdots + x^9)}_{x^{a_3}}\underbrace{(x^3 + \cdots + x^9)}_{x^{a_4}}$$

Choose $x^{a_1}$, $x^{a_2}$, $x^{a_3}$, $x^{a_4}$

$a_1 + a_2 + a_3 + a_4 = 20$ , $3 \leq a_i \leq 9$

$\underbrace{(a_1 - 3)}_{= b_1} + \underbrace{(a_2 - 3)}_{= b_2} + \underbrace{(a_3 - 3)}_{= b_3} + \underbrace{(a_4 - 3)}_{= b_4} = 8$ , $0 \leq b_i \leq 6$

$\binom{11}{3}$ choices of $b_1, .., b_4 \geq 0$ with $b_1 + b_2 + b_3 + b_4 = 8$

If $b_1 \geq 7$, then $\underbrace{(b_1 - 7)}_{\geq 0} + \underbrace{b_2}_{\geq 0} + \underbrace{b_3}_{\geq 0} + \underbrace{b_4}_{\geq 0} = 1$ $\binom{4}{3}$ choices

Similar for $b_2, b_3, b_4$. So the answer is

$$\binom{11}{3} - 4 \cdot \binom{4}{3} = 149.$$

**5. (8 pts)** Four pairs of twins are to be seated in a row for a photo so that no pair of twins can sit next to each other. In how many ways it can be done ?

Permutations of 8 people, $8!$.

Conditions $c_1, c_2, c_3, c_4$ : the corresponding pair of twins sit together.

$$N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) = N - \sum_{i=1}^{4} N(c_i) + \sum_{1 \leq i < j \leq 4} N(c_i c_j) - \sum_{1 \leq i < j < k \leq 4} N(c_i c_j c_k) + N(c_1 c_2 c_3 c_4)$$

$N(c_i) = 2 \cdot 7!$ : $A_1A_2$ and $A_2A_1$ 2 ways to sit together for twins.
$7 = 6$ remaining people $+ 1$ pair of twins together

$N(c_i c_j) = 2^2 \cdot 6!$ $N(c_i c_j c_k) = 2^3 \cdot 5!$

Answer: $8! - \binom{4}{1} \cdot 2 \cdot 7! + \binom{4}{2} \cdot 4 \cdot 6! - \binom{4}{3} \cdot 8 \cdot 5! + 16 \cdot 4!$