SOLUTIONS (Week 10)

1) $\frac{9}{35}$.

2) Sample space is the set of all possible choices of 3 cells out of 64 cells. Thus, size of the sample space is $C(64,3)$. In $C(8,3)\cdot P(8,3)$ different ways 6 checkers can be located such that no row or no column contains more than one checker. Then, the required probability is $\frac{C(8,3)\cdot P(8,3)}{C(64,3)} \approx 0.4516$.

3) Available odd numbers greater than 8 are 9 and 11. We can have the sum 9 with probability 4/36 and the sum with probability 2/36. Since the events of having a sum of 9 and 11 are disjoint, the probability of having an odd sum greater than 8 is 1/6.

4) Size of the sample space is $C(52,3)$ which consists of all possible choices of three cards out of 52 cards. We can have three cards none of which is a club in $C(39,3)$ different ways. Then, the probability of having at least on clubs is $1-\frac{C(39,3)}{C(53,3)} = 0.4135$.

5) $\frac{3}{6}\cdot\frac{2}{6}=\frac{1}{6}$.

6) We have to consider two events:

The first card is a face card of diamonds and the second card is a clubs.

The first card is a face card of clubs and the second card is another clubs.

Probabilities of these events are $\frac{3}{52}\cdot\frac{13}{51}=\frac{39}{2652}$ and $\frac{3}{52}\cdot\frac{12}{51}=\frac{36}{2652}$, respectively. Then the answer is $\frac{75}{2652}=0.02828\cdots$.

7) The product is a prime only when one die is a 1 and the other is a 2, 3 or 5. Then the probability is $\frac{6}{36}=\frac{1}{6}$.

8) $\frac{1}{6}$.

9) $1-\frac{49}{56}\cdot\frac{48}{55}=\frac{13}{55}$.

10) $\frac{16}{52}$ / $1-\frac{36}{52}\cdot\frac{35}{51}=\frac{116}{221}=0.52488\cdots$.

11) $\frac{21!}{26!}=0.0000001267\cdots$.

12) $\Pr(\text{No Ace}) = C(48,5)/C(52,5)$

$\Pr(\text{No K/Q}) = C(44,5)/C(52,5)$

$\Pr(\text{No Ace and No K/Q}) = C(40,5)/C(52,5)$

Then by principle of inclusion-exclusion the probability we are asked is $1-\frac{C(48,5)+C(44,5)-C(40,5)}{C(52,5)} = 0.1764\cdots$.

13) Sample space consists of all possible orderings of picked fruits (such as aaaooaaoaaoo), hence the size of sample space is $C(12,5)$. The last one is an apple in $C(11,5)$ orderings. Hence the probability we are asked is $\frac{C(11,5)}{C(12,5)}=0.5833\cdots$.

Alternative solution: The last fruit is an apple with probability $\frac{5}{12}$.

14) **a)** Five people can choose the floors they exit in 10^5 different ways. In $P(10,5)$ of these ways, choices are all distinct. The probability they all choose different floors is then $\frac{P(10,5)}{10^5}=0.3024$.

Alternative solution: Each floor is chosen with probability 10^{-1}, then five distinct floors (in a specific order) can be chosen in 10^{-5} ways. Number of choices is $C(10,5)$ and number of orderings is 5!. Then the probability that each one chooses a different floor is $10^{-5}\cdot C(10,5)\cdot 5! = 0{,}3024$.

b) Floor 10 is chosen with probability 2^{-1} and each other floor is chosen with probability 18^{-1},

-If no one chooses 10th floor, the probability that each one chooses a different floor is $18^{-5}\cdot C(9,5)\cdot 5!$,

-If one chooses 10th floor, they choose different floors with the probability $2^{-1}\cdot 18^{-4}\cdot C(9,4)\cdot 5!$.

Then, the required probability is $2^{-5}9^{-4}5!\left(9^{-1}C(9,5)+C(9,4)\right) = 18^{-5}\cdot 5!\cdot 10\cdot C(9,5)=\frac{525}{6561}=0.0800\cdots$.

15) We consider all possible cases:

- All the balls go to a unique box with probability 1/9, (a ball is thrown to any of the boxes, and then each of the others go to the same box with probability 1/3),
- one ball goes to each box (with probability 2/9),
- two balls go to a box, a ball goes to another box and a box is left empty (with probability 6/9),

16) $\frac{4}{15}$

17) $\frac{8!3!}{10!}=\frac{1}{15}$.

18) $\frac{D_n}{n!}\approx\frac{1}{e}$

19) Denote the people by $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2$ and define the events

A: A_1 and A_2 are in the same team,

B: B_1 and B are in the same team,

C: C_1 and C_2 are in the same team,

D: D_1 and D_2 are in the same team,

E: E_1 and E_2 are in the same team,

F: F_1 *and* F_2 are in the same team.

Then, the probability that none of the teams has a married couple is given by $p=\Pr(\bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D}\cap\bar{E}\cap\bar{F})$.

Now we compute the necessary probabilities:

$$\Pr(A)=\frac{1}{4}$$
$$\Pr(A\cap B)=\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{3}{64}$$
$$\Pr(A\cap B\cap C)=\frac{3}{4}\cdot\frac{2}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{3}{512}$$
$$\Pr(A\cap B\cap C\cap D)=\frac{3}{4}\cdot\frac{2}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{3}{2048}$$

Then by principle of inclusion-exclusion:

$$p=1-\binom{6}{1}\frac{1}{4}+\binom{6}{2}\frac{3}{64}-\binom{6}{3}\frac{3}{512}+\binom{6}{4}\frac{3}{2048}$$
$$=\frac{221}{2048}=0.1079\cdots.$$

20) $\frac{10}{35}$

21) $\frac{8}{35}$

22) $\frac{1}{10}$

23) $\frac{1}{15}$ (compare to problem 17.)

24) $\frac{C(5,2)}{\frac{1}{2}C(10,5)} = \frac{10}{126} = 0.0793\cdots$.

25) A function from the set of students to the set $\{1,2,3,4\}$ is defined and we are asked to find the probability that the function is not onto. The number of onto functions is given by

$$A = \sum_{k=0}^{4} (-1)^k \binom{4}{k} (4-k)^{10} = 818520.$$

The required probability is $1 - \frac{818520}{4^{10}} = 0.2193\cdots$.

26) There are $\frac{8!}{2!^4 4!} = 105$ ways to determine the pairs. If two fixed English teams are to be matched, matching the remaining teams can be completed in $\frac{6!}{2!^3 3!} = 15$ ways. Thus, in 45 different pairings, two English teams are in the same pair. Then, probability that no two English teams are matched is $1 - \frac{45}{105} = \frac{4}{7}$.

27) Let the pair (w, b) denote the number of white and black balls. From the situation we can pass to $(w-1, b+1)$ with probability $\frac{w}{5}$ or $(w+1, b-1)$ with probability $\frac{b}{5}$. Starting from $(3,2)$, the situation $(0,5)$ can be achieved in at most 5 steps in one of the following four ways:

$$(3,2) \xrightarrow{3/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{1/5} (0,5)$$
$$(3,2) \xrightarrow{2/5} (4,2) \xrightarrow{4/5} (3,2) \xrightarrow{3/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{1/5} (0,5)$$
$$(3,2) \xrightarrow{3/5} (2,3) \xrightarrow{3/5} (3,2) \xrightarrow{3/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{1/5} (0,5)$$
$$(3,2) \xrightarrow{3/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{4/5} (2,3) \xrightarrow{2/5} (1,4) \xrightarrow{1/5} (0,5)$$

Then the probability we are trying to compute is $\frac{6}{125} + \frac{48+54+48}{3125} = \frac{12}{125} = 0{,}096$.

28) First student cannot be the winner. Second one wins with the probability $\frac{9}{9}\cdot\frac{1}{9}$, third one has the probability $\frac{9}{9}\cdot\frac{8}{9}\cdot\frac{2}{9}$ to win, fourth one wins with the probability $\frac{9}{9}\cdot\frac{8}{9}\cdot\frac{7}{9}\cdot\frac{3}{9}, \ldots$. In general, k-th one wins with the probability $\frac{P(9,k-1)\cdot(k-1)}{9^k}$.

Computed values are as follows:

No	Probability to win
1	0,00000
2	0,11111
3	0,19753
4	0,23045
5	0,20485
6	0,14225
7	0,07587
8	0,02950
9	0,00749
10	0,00094

Fourth student has the largest probability to win the bonus.

29) **a)** Let `BWWWWWWWWW` denote the ordering of shuffled cards where the first card is black. X wins if the ordering is `BWWWWWWWWW` or `WWWBWWWWWW` or `WWWWWWBWW`. Second player wins for the orderings `WBWWWWWWWW` or `WWWWBWWWWW` or `WWWWWWWBW`. It follows that each player has the same probability $\frac{1}{3}$ to be the winner.

b) 2 black and 7 white cards can be arranged in $\frac{9!}{2!7!} = 36$ different ways. For the first player to win, the possible orderings are

`B [BWWWWWWW]`: 8 orderings

`WWWB [BWWWW]`: 5 orderings

`WWWWWWB [BW]`: 2 orderings

Winning orderings for the second are:

`WB [BWWWWWW]`: 7 orderings

`WWWWB [BWWW]`: 4 orderings

`WWWWWWWB [B]`: 1 ordering

Then, the probability that the first player wins is 15/36. Second player is the winner with probability 11/36 and consequently third one is the winner with probability 10/36.

30) Let $q = 1 - p$ and let P be the probability that Selim is the winner.

$$\begin{aligned} P &= p(p + qp^2 + qpqp^2 + qpqpqp^2 + \cdots) \\ &\qquad + q(p^2 + pqp^2 + pqpqp^2 \\ &\qquad + pqpqpqp^2 + \cdots) \\ &= (p^2 + qp^3 + q^2p^4 + \cdots) \\ &\qquad + q(p^2 + qp^3 + q^2p^4 + \cdots) \\ &= (1+q)p^2(1 + pq + p^2q^2 + p^3q^3 + \cdots) \\ &= \frac{(1+q)p^2}{1 - pq} \\ &= \frac{(2-p)p^2}{1 - p + p^2}. \end{aligned}$$