SOLUTIONS (Week 11)

1) **a)** $\binom{10}{2}\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^8 = 0.0746\cdots$.

b) $1-\binom{10}{0}\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{10}-\binom{10}{1}\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^9 = 0.0861\cdots$.

c) $\binom{10}{0}\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{10} = 0.5987\cdots$.

2) **a)** $\frac{1}{6}$ **d)** $1-\left(\frac{5}{6}\right)^{10} = 0.8384\cdots$

b) $\frac{5}{6}\cdot\frac{1}{6}=\frac{5}{36}$ **e)** $\left(\frac{5}{6}\right)^{k-1}\cdot\frac{1}{6}$

c) $\frac{1}{6}+\frac{5}{36}=\frac{11}{36}$ **f)** $\sum_{k=6}^{\infty}\left(\frac{5}{6}\right)^{k-1}\frac{1}{6}=\left(\frac{5}{6}\right)^5 = 0.4018\cdots$

3) $\frac{P(26,5)}{5^{26}} = 0.6643\cdots$

4) **a)** $(.72)^2 = 0.5184$ **b)** $(.28)^2 = 0.0784$

5) **a)** $\left(\frac{3}{10}\right)^3 = 0.027$ **d)** $\left(\frac{9}{10}\right)^3 = 0.729$

b) $\left(\frac{5}{10}\right)^3 = 0.125$ **e)** $1-\left(\frac{4}{10}\right)^3 = 0.936$

c) $\left(\frac{1}{10}\right)^3 = 0.001$

6) **a)** $\binom{20}{5}(0.08)^5(0.92)^{15} = 0.0145\cdots$

b) $\binom{20}{5}(0.08)^0(0.92)^{20} = 0.1886\cdots$

c) $\binom{20}{5}(0.08)^{20}(0.92)^0 = 1.15\cdot 10^{-22}$

7) $1-(0.999)^{224} = 0.2007\cdots$

8) $\binom{35}{4}(0.03)^4(0.97)^{31} = 0.0164\cdots$

9) $\sum_{i=1}^{5}\binom{5}{i}(0.007)^i(0.993)^{5-i} = 0.0345\cdots$ /

$\binom{5}{0}(0.007)^0(0.993)^5+\binom{5}{1}(0.007)^1(0.993)^4 = 0.9995\cdots$

10) $1-\binom{12}{0}(0.15)^0(0.85)^{12}-\binom{12}{1}(0.15)^1(0.85)^{11} = 0.5565\cdots$.

11) $\left(\frac{r}{6}\right)^6 = \frac{r^6}{46{,}656}$.

12) There are 3 correct and 7 fake keys in the box. One can pick two correct and one fake key in $\binom{3}{2}\binom{7}{1}=21$ possible ways. Choosing 3 correct keys is possible only in 1 way. Thus, there are 22 possible choices which enable us to open the door. Since the number of all possible choices is $\binom{10}{3}=120$, probability of opening the door is $\frac{21}{120}=\frac{7}{40}$.

13) **First solution.** Assume that the boy stops at the X th try. Then $Pr(X=k)=\left(\frac{4}{5}\right)^{k-1}\left(\frac{1}{5}\right)$ and

$$\Pr(X\le k)=\sum_{r=1}^{k}\left(\frac{4}{5}\right)^{r-1}\left(\frac{1}{5}\right)=\frac{1}{5}\sum_{r=0}^{k-1}\left(\frac{4}{5}\right)^{r}$$
$$=1-\left(\frac{4}{5}\right)^{k}.$$

Second solution. The probability of drawing a white ball in each of the first k tries is $\left(\frac{4}{5}\right)^k$. Consequently, to have a black ball in one of the first k tries is $1-\left(\frac{4}{5}\right)^k$.

14) **First Solution.** Assume that the boy stops at the X th try. We have

$$\Pr(X=1)=\frac{1}{3}$$
$$\Pr(X=2)=\frac{2}{3}\cdot\frac{1}{4}=\frac{2}{3\cdot 4}$$
$$\Pr(X=3)=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{1}{5}=\frac{2}{4\cdot 5}$$
$$\Pr(X=4)=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{1}{6}=\frac{2}{5\cdot 6}$$
$$\vdots$$
$$\Pr(X=k)=\frac{2}{(k+1)(k+2)}$$

Then

$$\Pr(X\le k)=\sum_{r=1}^{k}\frac{2}{(r+1)(r+2)}$$
$$=2\left[\sum_{r=1}^{k}\frac{1}{r+1}-\sum_{r=1}^{k}\frac{1}{r+2}\right]$$
$$=2\left[\sum_{r=1}^{k}\frac{1}{r+1}-\sum_{r=2}^{k+1}\frac{1}{r+1}\right]$$
$$=2\left(\frac{1}{2}-\frac{1}{k+2}\right)$$
$$=\frac{k}{k+2}.$$

Second solution. The probability of drawing a white ball in each of the first k tries is $\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdots\frac{k+1}{k+2}=\frac{2}{k+2}$. Consequently, to have a white ball in one of the first k tries is $1-\frac{2}{k+2}=\frac{k}{k+2}$.

15) $\Pr(\text{Bill wins})=\frac{7}{11}\cdot\frac{4}{10}+\frac{7}{11}\frac{6}{10}\frac{5}{9}\cdot\frac{4}{8}+\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\cdot\frac{4}{6}+\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}\cdot\frac{4}{4}=0.3939\cdots$.

16) $\sum_{i=20}^{30}\binom{30}{i}(0.8)^i(0.2)^{30-i} = 0.9744\cdots$

17) $\binom{8}{3}\left(\frac{3}{8}\right)^3\left(\frac{5}{8}\right)^5 = 0.2816\cdots$.

18) $\sum_{i=4}^{500}\binom{500}{i}(0.01)^i(0.99)^{500-i} = 1-\sum_{i=0}^{3}\binom{500}{i}(0.01)^i(0.99)^{500-i} = 0.7363\cdots$

19) Let x be the probability of having heads in a single flipping. When the coin is flipped 6 times the probability of 3 heads and 3 tails is $P(x)=\binom{6}{3}x^3(1-x)^3$. We have $P'(x)=60x^2(1-x^2)(1-2x)$. It follows that P attains its maximum value for $x=1/2$ and this maximum value is $P\left(\frac{1}{2}\right)=0.3125$. We conclude that, the probability of having 3 heads and 3 tails can never exceed 0.3125.

20) Let $P(n,k)$ denote the probability of having exactly k heads when the coin is flipped n times. Then

$$P(4,2)=\binom{4}{2}x^2(1-x)^2=6x^2(1-x)^2=0.24$$

From which we get $x(1-x)=0.2$ and

$$P(6,3)=\binom{6}{3}x^3(1-x)^3=20x^3(1-x)^3$$
$$=20[x(1-x)]^3=0.16\ .$$

21) $P(10,5)=\binom{10}{5}\cdot 2^{-10}=0.2460\cdots$ and

$P(20,10)=\binom{20}{10}\cdot 2^{-20}=0.1761\cdots$.