METU
DEPARTMENT OF MATHEMATICS

Math 112 Discrete Mathematics

Exercises 7

1) The sequence $\{a_n\}$ is defined by $a_n = a_{n-1} + n \cdot n!$ for $n \geq 1$. If $a_0 = 0$, find a_{112}.

[Hint. Simplify $(n+1)! - n!$.]

2) The sequence $\{a_n\}$ is defined by $a_{n+3} = a_{n+2} \cdot a_n - a_{n+1}$ for $n \geq 0$ and $a_0 = 1$, $a_1 = 2$ and $a_2 = 4$. Find a_{112}.

[Hint. Compute first ten terms.]

3) A sequence $\{a_n\}$ is defined by $a_{n+2} = a_{n+1} + na_n$ for $n \geq 0$. If $a_0 = 0$ and $a_7 = 38$, find a_5.

4) Given the recurrence relation $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$. Show that each of the following sequences satisfy this relation.

a) $1, 1, 1, 1, \dots$
b) $2, 2, 2, 2, \dots$
c) $1, 2, 4, 8, \dots, 2^n, \dots$
d) $1,3,9,27, \dots, 3^n, \dots$
e) $7,14,34,92, \dots, 2^n + 3^{n+1} + 3, \dots$

5) Find (other than the ones mentioned in the previous question) a sequence which satisfies the recurrence relation $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$.

6) Show that the sequence $1, 3, 9, 27, \dots, 3^n, \dots$ satisfies all of the following recurrence relations.

a) $a_n = 3a_{n-1}$,
b) $a_n = 5a_{n-1} - 6a_{n-2}$,
c) $a_n = 4a_{n-1} - 3a_{n-2}$,
d) $a_n = a_{n-1} + 6a_{n-2}$,
e) $a_n = -a_{n-1} + 9a_{n-2} + 9a_{n-3}$,
f) $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$.

7) Find (other than the ones mentioned in the previous question) a second order recurrence relation which is satisfied by the sequence $1, 3, 9, 27, \dots, 3^n, \dots$.

8) In each of the following, show that the given sequence satisfies the corresponding recurrence relation

a) $1, 2, 3, \dots, n, \dots$	$a_n = a_{n-1} + 1$	$n \geq 1$
b) $1, 2, 3, \dots, n, \dots$	$a_n = 2a_{n-1} - a_{n-2}$	$n \geq 2$
c) $0, 1, 3, 7, \dots, 2^n - 1, \dots$	$a_n = 3a_{n-1} - 2a_{n-2}$	$n \geq 2$
d) $1, 3, 5, 7, \dots, 2n - 1, \dots$	$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$	$n \geq 3$
e) $2, 4, 6, 8, \dots, 2n, \dots$	$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$	$n \geq 3$
f) $3,6,9,12, \dots, 3n, \dots$	$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$	$n \geq 3$
g) $0,1, 4, 9, \dots, n^2, \dots$	$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$	$n \geq 3$

9) Find the general solutions of the following recursions

a) $a_{n+3} = 8a_{n+2} - 19a_{n+1} + 12a_n$,
b) $a_{n+3} = 11a_{n+2} - 32a_{n+1} + 28a_n$,
c) $a_{n+3} = 6a_{n+2} - 12a_{n+1} + 8a_n$,

d) $a_{n+3} = 4a_{n+2} - 6a_{n+1} + 4a_n$.

10) Find the general term of the sequence $\{a_n\}$ if $a_0 = 1, a_1 = 2, a_2 = 3$ and for $n \geq 3$

a) $a_n = 7a_{n-2} - 6a_{n-3}$,
b) $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$,
c) $a_n = 9a_{n-1} - 27a_{n-2} + 27a_{n-3}$.

11) In each of the following, a sequence $\{a_n\}$ is defined recursively. Find an equivalent homogeneous recursive relation and provide sufficient initial conditions to define the same sequence

a) $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2} + 1$ for $n \geq 2$,
b) $a_0 = 1,\ a_1 = 1$ and $a_n = 2a_{n-1} - a_{n-2} + n$ for $n \geq 2$,
c) $a_0 = 2,\ a_1 = 1$ and $a_n = 3a_{n-1} - a_{n-2} + n^2 + 3$ for $n \geq 2$,
d) $a_0 = 2,\ a_1 = 7$ and $a_n = 2a_{n-1} - a_{n-2} + n$ for $n \geq 2$,
e) $a_0 = 3,\ a_1 = 4$ and $a_n = 2a_{n-1} - a_{n-2} + 7^n$ for $n \geq 2$,
f) $a_0 = 1,\ a = 1$ and $a_n = a_{n-1} - 3a_{n-2} + 3^n - n^2$ for $n \geq 2$,
g) $a_0 = 1,\ a = 1$ and $a_n = 2a_{n-1} + 4a_{n-2} + n2^n$ for $n \geq 2$.

12) If the sequence a_0, a_1, a_2, a_3 ... satisfies the relation $u_{n+2} = \alpha u_{n+1} + \beta u_n$, show that the sequence $a_0, a_2, a_4, a_6, \ldots$ satisfies the relation $v_{n+2} = (2\beta + \alpha^2)v_{n+1} - \beta^2 v_n$.

13) If the linear complexities of the sequences $\{a_n\}$ and $\{b_n\}$ are 2, show that the linear complexity of the sequence $\{a_n b_n\}$ is at most 4.

[Note. Linear complexity of a sequence is order of the smallest order constant coefficient linear homogeneous recursive relation satisfied by the sequence.]

14) It is given that the sequences $\{a_n\}$ and $\{b_n\}$ both satisfy the same constant coefficient linear homogeneous recursive relation of order 5. If $b_n = 3 \cdot 2^{n+1} + i^n + 1$ for any nonnegative integer n and $a_0 = 0,\ a_1 = 1,\ a_2 = 3, a_3 = 5, a_4 = 7$, find a_5.

15) If the sequences $\{a_n\}$ and $\{b_n\}$satisfy the relations

$$a_n = \alpha a_{n-1} + \beta b_{n-1}$$
$$b_n = \gamma a_{n-1} + \delta b_{n-1}$$

show that they both satisfy the relation

$$u_n = (\alpha + \delta)u_{n-1} + (\gamma\beta - \alpha\delta)u_{n-2}.$$

16) Let the sequence $\{a_n\}$ be defined by the relation $a_{n+2} = \alpha a_{n+1} + \beta a_n$. Find a constant coefficient linear homogeneous recursive relation which is satisfied by the sequence of partial sums $\{S_n\}$.

[Note. $S_n = a_0 + a_1 + \cdots + a_n$.]