SOLUTIONS (Week 7)

1) Using the hint, we get $(n+1)! - n! = n \cdot n!$. Then

$$\begin{aligned} a_1 &= a_0 + 2! - 1! \\ a_2 &= a_1 + 3! - 2! \\ a_3 &= a_2 + 4! - 3! \\ &\vdots \\ a_{n-1} &= a_{n-2} + n! - (n-1)! \\ a_n &= a_{n-1} + (n+1)! - n! \end{aligned}$$

If we add all the expressions side by side, we get $a_n = (n+1)! - a_0 + 1! = (n+1)! + 1$. Then $a_{112} = 112! + 1$.

2) First ten terms are $1, 2, 4, \mathbf{2}, \mathbf{0}, \mathbf{-2}, -4, \mathbf{2}, \mathbf{0}, \mathbf{-2}$. The pattern $2, 0, -2$ is repeated. Since the order of recursion is 3, we conclude that after the fourth term (a_3), the sequence has period 4. Then, $a_{112} = a_4 = 0$.

3)

$a_2 = a_1$
$a_3 = a_2 + a_1 = 2a_1$
$a_4 = a_3 + 2a_2 = 4a_1$
$a_5 = a_4 + 3a_3 = 10a_1$
$a_6 = a_5 + 4a_4 = 26a_1$
$a_7 = a_6 + 5a_5 = 76a_1$

Then $a_1 = \frac{1}{2}$.

4)

a) $1, 1, 1, 1, \ldots$
b) $2, 2, 2, 2, \ldots$
c) $1, 2, 4, 8, \ldots, 2^n, \ldots$
d) $1,3,9,27, \ldots, 3^n, \ldots$
e) $7,14,34,92, \ldots, 2^n + 3^{n+1} + 3, \ldots$

5) For any choice of the real numbers A, B and C, the sequence with the general term $A2^n + B3^n + C$ satisfies the given recursion.

6)

a) $3^n = 3 \cdot 3^{n-1}$.
b) $5 \cdot 3^{n-1} - 6 \cdot 3^{n-2} = 5 \cdot 3^{n-1} - 2 \cdot 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.
c) $4 \cdot 3^{n-1} - 3 \cdot 3^{n-2} = 4 \cdot 3^{n-1} - 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.
d) $3^{n-1} + 6 \cdot 3^{n-2} = 3^{n-1} + 2 \cdot 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.
e) $-3^{n-1} + 9 \cdot 3^{n-2} + 9 \cdot 3^{n-3} = -9 \cdot 3^{n-3} + 3^n + 9 \cdot 3^{n-3} = 3^n$.
f) $3^{n-3} + 15 \cdot 3^{n-3} - 6 \cdot 3^{n-3} = 27 \cdot 3^{n-3} = 3^n$.

7) For any $\lambda \in \mathbb{R}$, the sequence $\{3^n\}$ satisfies the relation $a_{n+2} = (3-\lambda)a_{n+1} + 3\lambda a_n$ for $n \geq 0$.

8)

a) $(n-1) + 1 = n$.
b) $2(n-1) - (n-2) = n$.
c) $3(2^{n-1} - 1) - 2(2^{n-2} - 1) = 6 \cdot 2^{n-2} - 2 \cdot 2^{n-2} - 1 = 2^n - 1$.
d) $3[2(n-1) - 1] - 3[2(n-2) - 1] + [2(n-3) - 1] = 3[2n-3] - 3[2n-5] + [2n-7] = 2n - 1$.
e) $3(2n-2) - 3(2n-4) + (2n-6) = 2n$.
f) $3(3n-3) - 3(3n-6) + (3n-9) = 3n$.
g) $3(n-1)^2 - 3(n-2)^2 + (n-3)^2 = 3(n^2 - 2n + 1) - 3(n^2 - 4n + 4) + (n^2 - 6n + 9) = n^2$.

9)

a) $a_n = A + B3^n + C4^n$.

b) $a_n = A2^n + Bn2^n + C7^n$.

c) $a_n = A2^n + Bn2^n + Cn^2 2^n$.

d) $a_n = A2^n + B(1-i)^n + C(1+i)^n$.

10)

a) $a_n = \frac{1}{20}(5 + 2^{n+4} - (-3)^n)$.

b) $a_n = 2^{n+1} + n2^{n-1} - 3^n$.

c) $a_n = 3^n - n3^{n-1}$.

11)

a) $a_0 = a_1 = 1,\ a_2 = 3$ and $a_n = 2a_{n-1} - a_{n-3}$ for $n \geq 3$.

b) $a_0 = a_1 = a_2 = a_3 = 1$ and $a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}$ for $n \geq 4$.

c) $a_0 = 2,\ a_1 = 1,\ a_2 = 8,\ a_3 = 34,\ a_4 = 113$ and $a_n = 6a_{n-1} - 13a_{n-2} + 13a_{n-3} - 6a_{n-4} + a_{n-5}$ for $n \geq 5$.

d) $a_0 = 2,\ a_1 = 7,\ a_2 = 12,\ a_3 = 17$ and $a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}$ for $n \geq 4$.

e) $a_0 = 3,\ a_1 = 4,\ a_2 = 54,$ and $a_n = 9a_{n-1} - 15a_{n-2} + 7a_{n-3}$ for $n \geq 3$.

f) $a_0 = 1,\ a = 1$ and $a_n = a_{n-1} - 3a_{n-2} + 3^n - n^2$ for $n \geq 2$,

g) $a_0 = 1,\ a = 1$ and $a_n = 2a_{n-1} + 4a_{n-2} + n2^n$ for $n \geq 2$.

12) We have to show that $a_{2(n+2)} = (2\beta + \alpha^2)a_{2(n+1)} - \beta^2 a_{2n}$.

$$\begin{aligned} a_{2n+4} &= \alpha a_{2n+3} + \beta a_{2n+2} \\ &= \alpha[\alpha a_{2n+2} + \beta a_{2n+1}] + \beta a_{2n+2} \\ &= (\alpha^2 + \beta)a_{2n+2} + \alpha\beta a_{2n+1} \\ &= (\alpha^2 + \beta)a_{2n+2} + \beta(a_{2n+2} - \beta a_{2n}) \\ &= (\alpha^2 + 2\beta)a_{2n+2} - \beta^2 a_{2n}. \end{aligned}$$

13) Let $a_{n+2} = \alpha a_{n+1} + \beta a_n$ and $b_{n+2} = \gamma b_{n+1} + \delta b_n$ for some $\alpha, \beta, \gamma, \delta \epsilon \mathbb{R}$ and $n \geq 0$. We first assume that characteristic equations $r^2 - \alpha r - \beta = 0$ and $s^2 - \gamma s - \delta = 0$ both have two distinct roots say r_1, r_2 and s_1, s_2. Then general terms of the sequences are $a_n = Ar_1^n + Br_2^n$ and $b_n = Cs_1^n + Ds_2^n$ for some $A, B, C, D \in \mathbb{R}$. It follows that the general term of $\{a_n b_n\}$ is $AC(r_1 s_1)^n + AD(r_1 s_2)^n + BC(r_2 s_1)^n + BD(r_2 s_2)^n$. It is obvious that the sequence $\{a_n b_n\}$ satisfies a constant coefficient linear homogeneous recursive relation of order whose characteristic equation is $(t - r_1 s_1)(t - r_1 s_2)(t - r_2 s_1)(t - r_2 s_2) = 0$. We conclude that the linear complexity of $\{a_n b_n\}$ is at most 4.

The case of the multiple roots can be shown similarly.

14) Characteristic equation of the common recursion is $(r-2)^2(r^2+1)(r-1) = r^5 - 5r^4 + 9r^3 - 9r^2 + 8r - 4 = 0$. Then $a_{n+5} = 5a_{n+4} - 9a_{n+3} + 9a_{n+2} - 8a_{n+1} + 4a_n$ and in particular $a_5 = 5a_4 - 9a_3 + 9a_2 - 8a_1 + 4a_0 = 9$.

15) By putting $b_{n-1} = \frac{1}{\beta}(a_n - \alpha a_{n-1})$ in $b_n = \gamma a_{n-1} + \delta b_{n-1}$, we get $a_{n+1} - \alpha a_n = \beta\gamma a_{n-1} + \delta a_n + \delta\alpha a_{n-1}$ which simplifies into $a_{n+1} = (\alpha + \delta)a_n + (\gamma\beta - \alpha\delta)a_{n-1}$. This proves that $\{a_n\}$ satisfies the given relation. In a similar way it can be shown that $\{b_n\}$ satisfies the same relation.

16) Since $S_n = S_{n-1} + a_n$ for any positive integer n, we can write

$$\beta\,\underbrace{(S_n - S_{n-1} - a_n)}_{=0} + \alpha\,\underbrace{(S_{n+1} - S_n - a_{n+1})}_{=0} - \underbrace{(S_{n+2} - S_{n+1} - a_{n+2})}_{=0} = 0.$$

Then $-\beta S_{n-1} + (\beta - \alpha)S_n + (\alpha + 1)S_{n+1} - S_{n+2} + \underbrace{(a_{n+2} - \alpha a_{n+1} - \beta a_n)}_{=0} = 0$ and finally,

$$S_{n+2} = (\alpha + 1)S_{n+1} + (\beta - \alpha)S_n - \beta S_{n-1}.$$