SOLUTIONS (Week 8)

1) Let A_n be the number of ways Ayşe can climb a staircase with n stairs. At the first step if she climbs one stair, she can climb the remaining $n-1$ stairs in A_{n-1} ways. If the first step of Ayşe is 2 stairs, there are A_{n-2} ways to climb the remaining part of the staircase. It follows that $A_n = A_{n-1} + A_{n-2}$. Since $A_1 = 1$ and $A_2 = 2$ we observe that $A_n = F_n$.

2) Call a subset of $A = \{1,2,\dots,n\}$ which do not contain any pair of successive integers an n −stramboshe and let S_n be the number of n −stramboshes. If an n −stramboshe X does not contain n, it can be regarded as an $(n-1)$-stramboshe. If it contains n, it can be written as $B \cup \{n\}$ where B is an $(n-2)$- stramboshe. It follows that $S_n = S_{n-1} + S_{n-2}$. Since $S_1 = 2$ (1-stramboshes are $\emptyset$ and $\{1\}$), $S_2 = 3$ (2-stramboshes are $\emptyset$, $\{1\}$, $\{2\}$) we see that S_n is the sequence 2 , 3 , 5 , 8 , …. Then, $S_n = F_{n+1}$.

3) Let B_n be the number of ways of tiling a $1 \times n$ rectangular board using 1×2 and 1×1 pieces. If the first piece is 1×1, then the remaining part can be tiled in B_{n-1} ways. If the first part is 1×2, remaining part can be tiled in B_{n-2}. Then, $B_n = B_{n-1} + B_{n-2}$ for $n \geq 2$ and $B_1 = 1, B_2 = 2$. We conclude that $B_n = F_n$.

4) Let C_n be the number of ways of tiling a $2 \times n$ rectangular board using 1×2 and 2×2 pieces. We can start the tiling in one of three ways: with a 'vertical' 1×2 piece, with a 2×2 piece or with two 'horizontal' 1×2 pieces (one on top of the other). In the first case, the remaining part can be tiled in C_{n-1} ways and in each of the other cases rest of the board can be tiled in C_{n-2} ways. Thus, $C_n = C_{n-1} + 2C_{n-2}$. Characteristic equation of the recursion is $r^2 - r - 2r = (r-2)(r+1) = 0$ whose roots are $r_1 = -1, r_2 = 2$. Then $C_n = A2^n + B(-1)^n$. Above discussion gives that $C_1 = 1$ and $C_2 = 3$. Using these initial values we have the system

$$\begin{aligned} 2A - B &= 1 \\ 4A + B &= 3 \end{aligned}$$

whose solution is $A = \frac{2}{3}, B = \frac{1}{3}$. Then $C_n = \frac{1}{3}(2^{n+1} + (-1)^n)$.

5) Let T_n be number of messages that can be transmitted in n seconds. It can be shown that $T_n = T_{n-1} + 2T_{n-2}$. As $T_1 = 1$ and $T_2 = 3$, from the previous problem it follows that $T_n = \frac{1}{3}(2^{n+1} + (-1)^n)$.

6) Let p_n be the number of such permutations. The last term of such a permutation can be either n or $n-1$. If the last term is n, there are p_{n-1} different arrangements for the first $n-1$ terms. If the last term is $n-1$, then the term next to the last one must be n. In this case first $n-2$ terms can be arranged in p_{n-2} distinct ways. It follows that $p_n = p_{n-1} + p_{n-2}$ for $n \geq 2$ and $p_1 = 1,\ p_2 = 2$. Then, $p_n = F_n$.

7) Let E_n be the number of strings of length n formed with letters A, B and C with an even number of A's. Then the number of strings with an odd number of A's is $3^n - E_n$. For a string of length n with an even number of A's there are two possibilities:

- The first letter is B or C. Then the remaining letters can be arranged in E_{n-1} different ways. By the product rule, there are $2E_{n-1}$ such strings.
- The first letter is A. The remaining letters can be arranged in $3^{n-1} - E_{n-1}$.

Consequently, $E_n = E_{n-1} + 3^{n-1}$. We can write $3E_{n-1} = 3E_{n-2} + 3^{n-1}$. From these two recursions we obtain a homogeneous recursion $E_n = 4E_{n-1} - 3E_{n-2}$ with characteristic equation is $(r-1)(r-3) = 0$. It follows that $E_n = A + B3^n$ for some $A, B \in \mathbb{R}$. Using the initial conditions $E_1 = 2, E_2 = 5$ we find $A = B = \frac{1}{2}$. Thus $E_n = \frac{3^n+1}{2}$.

8) Let u_n be the number of strings of upper case letters of length n that do not contain AA. By counting, $u_1 = 3, u_2 = 8$. Call a string valid if it does not contain AA. Consider a valid string of length n. There are two cases depending on whether the first letter is A.

Case 1. The first letter is not A (2 choices). Then, the remaining $n-1$ letters can be any valid string of length $n-1$. Since there u_{n-1} of these, by the Rule of Product there are $2u_{n-1}$ valid string in which the first letter is not A.
Case 2. The first letter is A (1 choice). Since the string is valid, the second letter is not A (2 choices), and then the remaining $n-2$ letters can be any valid string of length $n-2$. Since there u_{n-2} of these, by the product rule there are $2u_{n-2}$valid strings in which the first letter is A.
Therefore, by the addition rule $u_n = 2(u_{n-1} + u_{n-2})$.
Characteristic equation of the recurrence is $r^2 - 2r - 2 = 0$ whose roots are $r_{1,2} = 1 \pm \sqrt{3}$. Then general term is of the form $u_n = A(1+\sqrt{3})^n + B(1-\sqrt{3})^n$. Using the initial conditions we compute $A = \frac{3-7\sqrt{3}}{6}$ and $B = \frac{3+7\sqrt{3}}{6}$. Then $u_n = \frac{3-7\sqrt{3}}{6} \cdot (1-\sqrt{3})^n + \frac{3+7\sqrt{3}}{6} \cdot (1+\sqrt{3})^n$.

9) Imitating the solution of problem 7, we have the relation $E_n = 24E_{n-1} + 26^{n-1}$ from which the homogeneous relation $E_n = 24E_{n-1} + 624E_{n-2}$ is obtained. Then $E_n = A \cdot 24^n + B \cdot 26^n$ for some $A, B \in \mathbb{R}$. Using the initial conditions $E_1 = 25,\ E_2 = 626$ we compute $A = B = \frac{1}{2}$. Then $E_n = \frac{24^n + 26^n}{2}$.

10) Following the solution of problem 8 we obtain the relation $u_n = 25(u_{n-1} + u_{n-2})$ for $n \geq 3$ and the initial terms $u_1 = 26, u_2 = 675$. General term is $u_n = \frac{145-27\sqrt{29}}{290} \cdot \left(\frac{25-5\sqrt{29}}{2}\right)^n + \frac{145+27\sqrt{29}}{290} \cdot \left(\frac{25+5\sqrt{29}}{2}\right)^n$.

11) **a)** Let L_n be the largest possible number of regions that can be defined by n straight lines in plane. $L_1 = 2$ and L_n satisfies $L_n = L_{n-1} + n$ for $n \geq 2$. We obtain the homogeneous relation $L_n = 3L_{n-1} - 3L_{n-2} + L_{n-3}$ for $n \geq 3$ with the initial conditions $L_1 = 2,\ L_2 = 4,\ L_3 = 7$. General term is given by $L_n = \frac{1}{2}(n^2 + n + 1)$.

b) Let C_n be the largest possible number of regions that can be defined by n circles in plane. $C_1 = 2$ and C_n satisfies $C_n = C_{n-1} + 2(n-1)$ for $n \geq 2$. We obtain the homogeneous relation $C_n = 3C_{n-1} - 3C_{n-2} + C_{n-3}$ for $n \geq 3$ with the initial conditions $C_1 = 2,\ C_2 = 4,\ C_3 = 8$. General term is given by $C_n = n^2 - n + 2$.

c) Let T_n be the largest possible number of regions that can be defined by n triangles in plane. $T_1 = 2$ and T_n satisfies $T_n = T_{n-1} + 6(n-1)$ for $n \geq 1$. We obtain the homogeneous relation $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}$ for $n \geq 2$ with the initial conditions $T_1 = 2, T_2 = 8,\ T_3 = 26$. General term is given by $T_n = 6n^2 - 12n + 8$.

d) Let R_n be the largest possible number of regions that can be defined by n rectangles in plane. $R_1 = 2$ and R_n satisfies $R_n = R_{n-1} + 8(n-1)$ for $n \geq 1$. We obtain the homogeneous relation $R_n = 3R_{n-1} - 3R_{n-2} + R_{n-3}$ for $n \geq 3$ with the initial conditions $R_1 = 2,\ R_2 = 10,\ R_3 = 34$. General term is given by $R_n = 8n^2 - 16n + 10$.

12) For the number u_n of ways of paintings we can derive the recursion $u_n = 3 \cdot 2^n - u_{n-1}$ for $n \geq 2$ with the initial condition $u_1 = 0$. This recursion reduces to the homogeneous relation $u_n = u_{n-1} + 2u_{n-2}$ subject to $u_1 = 0$ and $u_2 = 6$. Roots of the characteristic equation $r^2 - r - 2 = (r-2)(r+1) = 0$ are $r_1 = -1$ and $r_2 = 2$. Then we get $u_n = 2^n + 2 \cdot (-1)^n$.

13) **a)** For the first column we have six possibilities: BY, BW, YB, YW, WB, WY. For any choice, we have 3 possibilities for the next column (for example if the first column is YW, second column can be BY, WY or WB). It follows that the entire board can be painted in $6 \cdot 3^{n-1} = 2 \cdot 3^n$ different ways.

b) Let A_n be the number of ways of painting the board so that no two white squares are adjacent. Also let W_n be the number of those paintings which has a white cell in the first column and put $R_n = A_n - W_n$. Then we can write $W_n = 4(A_{n-1} - W_{n-1}) + 2W_{n-1}$. This recurrence is equivalent with $A_n - R_n = 4R_{n-1} + 2(A_{n-1} - R_{n-1})$. Observing that $R_n = 4A_{n-1}$, we obtain $A_n - 4A_{n-1} = 16A_{n-2} + 2A_{n-1} - 8A_{n-2}$ which simplifies into $A_n = 6A_{n-1} + 8A_{n-2}$. Characteristic equation of the relation is $r^2 - 6r - 8 = 0$ whose roots are $r_{1,2} = 3 \mp \sqrt{17}$. Initial terms are $A_1 = 8$ and $A_2 = 56$. General term is $A_n = \frac{17-5\sqrt{17}}{34} \cdot \left(3 - \sqrt{17}\right)^n + \frac{17+5\sqrt{17}}{34} \cdot \left(3 + \sqrt{17}\right)^n$.