SOLUTIONS (Week 9)

1) **a)** Observe that there are only four pairs of integers in A that add up to 9, and that each integer exactly occurs in exactly one pair. These pairs are $1+8$, $2+7$, $3+6$, $4+5$. Let the pigeons be the five integers selected from A, and let the pigeonholes be the pairs of integers that add up to 9. According to pigeonhole principle, since there are more pigeons (5) than pigeonholes (4), at least two pigeons must go to the same hole. Thus, two distinct integers are sent to the same pair. But that implies that those two integers are the two distinct elements of the pair, so their sum is 9.

b) The answer is no. For instance consider the numbers: 1, 2, 3, 4.

2) Think of associating the married couples with boxes labeled 1 to n, so that whenever a member is selected from the set of $2n$ people, then that person is placed into his or her associated box. Thus the question reduces to asking for the smallest number of members that can be placed in the n boxes in order that some box contains two members. Clearly n does not suffice; however, by the pigeonhole principle, $n+1$ works.

3) The number of wins for a player is 1 or 2 or 3 … or $n-1$. These $n-1$ numbers correspond to $n-1$ pigeonholes in which the pigeons (players) are to be housed. So at least two of them should be in the same pigeonhole and they have the same number of wins.

4) **a)** 55 **b)** 52

5) In this example, the pigeons are the 150 people and the pigeonholes are the 29 possible last initials of their names (we consider the Turkish alphabet which consists of 29 letters). Note that $150 > 5 \times 28 = 140$. Then, the generalized pigeonhole principle states that some initial must be the image of at least six $(5+1)$ people. Thus at least six people have the same last initial.

6) Suppose that only four computers were used by three or more students. At most six students are allowed to share any computer, making a total of at most 24 students using these four computers. Since there are 42 students at all, that would leave at least 18 students to share the remaining eight computers with no more than two students per computer. But the generalized pigeonhole principle guarantees that if 18 students share eight computers, then at least three must share one of them. This is a contradiction. Thus the supposition is false, and so at least five computers are used by three or more students.

7) The least number of marbles to be picked is $(3-1)+(4-1)+(5-1)+1=10$.

8) Let $x_1, x_2, \dots, x_{1001}$ denote the 1001 integers chosen. We can express each x_i, $i = 1,2,\dots,1001$ in the form $x_i = 2^{m_i} b_i$ where b_i is odd (for instance, $564 = 2^2 \cdot 141$, $1184 = 2^5 \cdot 37$, $512 = 2^9 9 \cdot 1$, $97 = 2^0 \cdot 97$). Now, consider the odd integers $b_1, b_2, \dots, b_{1001}$. But, there are exactly 1000 odd integers between 1 and 2000, hence by the pigeonhole principle, at least two of $b_1, b_2, \dots, b_{1001}$ are equal. So $b_r = b_s$ for some r and s, where $r \neq s$. Then we have $x_r = 2^{m_r} b_r$ and $x_s = 2^{m_s} b_s$. Since $x_r \neq x_s$, we have $m_r \neq m_s$; without loss of generality we can assume $m_r < m_s$. But then $2m_r | 2m_s$, and consequently, $x_r | x_s$.

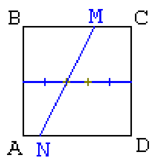
(Is it possible to choose 1000 integers from the list so that no number is divisible by any other?)

9) Number the locations $1,2,\dots,100$ and let a_i be the number assigned to location i, $i = 1,2,\dots,100$. There are 100 sums to consider: $a_1+a_2+a_3$, $a_2+a_3+a_4, \dots, a_{98}+a_{99}+a_{100}$, $a_{99}+a_{100}+a_1$, $a_{100}+a_1+a_2$ and each a_i appears in exactly three of the sums. Hence, the total of these sums is$3(a_1+a_2+\cdots+a_{100}) = 3 \cdot 5050 = 15150$. Thus, the average value of is $15150/100 = 151.5$, and so by the previous problem, one of the sums has value at least 152.

10) Number the days $1,2,\dots,12$ and consider the subsets. $\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\},\{11,12\}$. Since these subsets partition the $12-$day period, the 110 hours of practice can be distributed among them. And since $110 > 18 \times 6 = 108$, the generalized pigeonhole principle implies that some consecutive $2-$day period contains at least 19 hours.

11) Split the triangle into four smaller ones by connecting midpoints of its sides. The largest possible distance between two points of one small triangle is ½. Now, we are given 4 triangles and 5 points. By the Pigeonhole Principle, at least one triangle contains at least two points. The distance between any two such points does not exceed ½.

12) None of the given lines may pass through two successive sides of the square because in this case we get a triangle and a pentagon and not two quadrilaterals.



Assume one of them intersects sides BC and AD at points M and N, respectively. The quadrilaterals, $ABMN$ and $CDNM$, are both trapezoids with equal heights. Therefore, their areas are in the same ratio as their midlines. From here, MN intersects the midline of the square in ratio 2:3. This is true for any one of the nine lines. But there are only four points that divide the midlines of the square in the ratio 2:3. Therefore, by the Pigeonhole Principle, at least three of the lines pass through the same point.

13) The midpoint of the line joining two grid points (x_1, y_1) and (x_2, y_2) is located at $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. The latter will be a grid point iff its coordinates are integers. The x-coordinate will be integer iff x_1 and x_2 have the same parity, i.e., iff they are either both even or both odd. Out of 5 points, at least three satisfy this condition. But the same is true of the y-coordinate. And out of the selected three points, at least two have y-coordinate with the same parity.

14) In the following we assume $f(x)$ is a polynomial with integral coefficients:

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$$

Lemma. For any two different integers p and q, the difference $f(p)–f(q)$ is divisible by $p–q$.

Proof. Indeed, $f(p)–f(q) = c_n(p^n–q^n) + c_{n-1}(p^{n-1}–q^{n-1}) + \ldots + c_1(p - q)$ and, since $(p–q) \mid (p^k–q^k)$ for every integer $k > 0$.

Now, assume that $f(a) = f(b) = f(c) = 2$ and $f(d) = 3$ with all a, b, c and d different. From Lemma we immediately obtain that

$$(d–a) \mid (f(d)–f(a)) = 3–2 = 1,$$

$$(d–b) \mid (f(d)–f(b)) = 3–2 = 1,$$

$$(d–c) \mid (f(d)–f(c)) = 3–2 = 1.$$

Thus differences $d–a, d–b, d–c$ all divide 1. But 1 has only two divisors: 1 and -1. Therefore, by the Pigeonhole Principle, two of the differences coincide. Which contradicts our assumption that the numbers a, b, c are all different.

15) There are 1997 remainders of division by 1997. Consider the sequence of powers $1, 3, 3^2, \ldots, 3^{1997}$. It contains 1998 members. Therefore, by the Pigeonhole principle, some two of them, say $3n$ and $3m$, $n > m$, have equal remainders when divided by 1997. Then their difference $(3n–3m)$ is divisible by 1997.

16) As in previos problem, let 3^n and 3^m $(n > m)$ have the same remainder when divided by 1000. Thus $3^n–3^m = 3^m(3^{n-m}–1)$ is divisible by 1000. Since 1000 and $3m$ have no common factors, 1000 is bound to divide the second factor $(3^{n-m}–1)$. This exactly means that 3^{n-m} ends with 001.

17) Form the n consisting of the i –th odd integer less than $2n$ together with its multiples by powers of 2:

$$A_1 = \{1, 2, 4, , \ldots, 512\},$$
$$A_3 = \{3, 6, 12, 24, \ldots, 768\},$$
$$A_5 = \{5, 10, 20, \ldots, 640\},$$
$$\vdots$$
$$A_{999} = \{999\}.$$

Then the union of these n sets contains $\{1, 2, \ldots, 2n\}$. Hence some two of the selected integers belongs to A_i, for some i; and so one of them divides the other.

18) Let a_i be the total number of aspirin consumed up to and including the i –th day, for $i = 1, \ldots, 30$. Combine these with the numbers $a1 + 14, \ldots, a30 + 14$, providing 60 numbers, all positive and less or equal $45 + 14 = 59$. Hence two of these 60 numbers are identical. Since all a_i's and, hence, $(a_i + 14)$'s are distinct (at least one aspirin a day consumed), then $a_j = a_i + 14$, for some $i < j$. Thus, on days $i + 1$ to j, the person consumes exactly 14 aspirin.

19) Five women each cast in 3 plays makes 15 woman's parts in the 7 plays. Since $\frac{15}{7} > 2$, some play has at least 3 women in its cast.

20) A person at the party can have 0 up to $n - 1$ friends at the party. However, if someone has 0 friends at the party, then no one at the party has $n - 1$ friends at the party, and if someone has $n - 1$ friends at the party, then no one has 0 friends at the party. Hence the number of possibilities for the number of friends the n people at the party have must be less than n. Hence two people at the party have the same number of friends at the party.

21) No. Consider the set {3,4,5,6,7,8}

22) Yes. Any subset with 7 or more elements must contain both members of at least one of the following pairs:

(3,13), (4,12), (5,11), (6,10), (7,9)

23) Consider the partition:

$$\underbrace{(7,93), (8,92), \ldots, (49,51)}_{43\ pairs}, \underbrace{(50), (94), (95), (96), (97)}_{5-singletons}$$

24) A has 64 subsets and largest possible sum is $7 + 8 + 9 + 10 + 11 + 12 = 57$.

25) See Problem 10.

Problems and solutions are taken from the site: www.cut-the-knot.org